# SIMILARITY OF NON-NEWTONIAN FLOWS. III.* METZNER-RABINOWITSCH FLOWS

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Mathematical models of non-Newtonian flow are studied for a subgroup of viscometric, so called Metzner–Rabinowitsch, flows. It is shown here that the course of the velocity field can be estimated for these hydrodynamic situations on the basis of relations between macroscopic flow parameters $P_c$, $U_c$, $R_c$, without regard to the character of the non-Newtonian anomaly.

The Poiseuille flow, *i.e.* the laminar steady flow in a sufficiently long tube of circular cross-section[1–4] was one of the first hydrodynamic situations that were studied in relation with the non-Newtonian behaviour of some fluids also from the engineering point of view. For the Poiseuille flow the following relation for the value of velocity gradient on the wall was independently derived by a number of authors:

$$-\left.\frac{dv_z(r)}{dr}\right|_{r=R} = \frac{U_c}{R_c} \cdot \left(3 + \frac{d \ln (U_c/R_c)}{d \ln P_c}\right). \tag{1}$$

It has become customary to call it the Metzner–Rabinowitsch*** relation after the authors who have derived and applied it in rheometry[3] and in engineering correlations[4–6].

Beside the Poiseuille flow there exist other hydrodynamic situations for which similar relation can be derived and applied analogically, also for the approximative engineering correlations. Coleman, Markowitz and Noll[7] give relations of the type (*1*) for calculation of the deformation rate on the wall for Poisseuille, slit, and Couette flows. Skelland[8] as well gives the solution for the slit and considers the possibility of generalized correlations for this arrangement analogical to correlation postulated by Metzner for the Poisseuille flow. Kozicki with coworkers[9] propose generalized correlation for the flow through channels of arbitrary cross-sections, based on known relations of the type (*1*) for flow through the tube and slit. Lescarboura with coworkers[10] propose the procedure of approximative evaluation of rheometrical results for a number of arrangements, based again on the Metzner–Rabinowitsch Eq. (*1*) and on the known solutions of mathematical models of more general flow situations for the power law liquids. In our works[11,12] we derived exact relation of the type (*1*) for an annulus and applied it in a form of generalized correlations for the laminar heat transfer in an annulus[13].

With regard to possibilities of engineering application of the relations of the type (*1*) denoted as the Metzner–Rabinowitsch relations, we shall study the common

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*** Rabinowitsch[3] himself ascribes priority of derivation of relation (*1*) to K. Weissenberg.

structure of mathematical models of hydrodynamic situations which may be formulated by relations of the type (*1*). Among these hydrodynamic situations belong beside the Poiseuille flow, flow through the slit and the planar Couette flow, as we shall show, the rotation and translation of a cylinder in an infinite environment as well. We do not know any other arrangements which would admit to formulate the Metzner–Rabinowitsch equations. Nevertheless, we shall formulate basical properties of corresponding mathematical models in general to make obvious what is common for these hydrodynamic arrangements.

## Metzner–Rabinowitsch Flows

Each steady flow situation can be characterized by three macroscopic parameters[11,12,14] — characteristic velocity $U_c$, characteristic length of the system wall $R_c$ and by characteristic dynamic parameter $P_c$ (Table I). As Metzner–Rabinowitsch flows (further MR-flows) such hydrodynamic situations are considered for which it is possible: *1.* To define $P_c$ so that without regard to other flow conditions, *i.e.* liquid properties and parameters $R_c$ and $U_c$ (with the condition that the effect of inertia forces remains negligible) the relation

$$\tau_w = P_c \tag{2}$$

holds, where $\tau_w$ is that value of shear stress, which is constant on all walls of the system. *2.* To express without regard to other flow conditions the value of shear rate at the walls, $D_w$, in the form of Metzner–Rabinowitsch equation, *i.e.* as

$$D_w = U_c/R_c \,.\, \phi(n^*)\,, \tag{3}$$

where $n^*$ is a parameter dependent on flow conditions which can be determined either experimentally or numerically as

$$n^* = \frac{\mathrm{d}\ln P_c}{\mathrm{d}\ln (U_c/R_c)}\,. \tag{4}$$

Since we do not know so far, how to define exactly the kinematic or dynamic class of all MR-flows, similarly as the class of viscometric flows[7] is defined, we limit ourselves to presenting of some properties of mathematical models of MR-flows which are perhaps the result of a more general limitation and which are sufficient for derivation of MR-relations: *1.* MR-flows are a subgroup of viscometric flows, their kinematics can be taken as mutually sliding planes which themselves are not deformed. It is therefore possible to choose a system of orthogonal coordinates $x_1$, $x_2$, $x_3$ so that $x_1$ = const. is an analytical expression of the course of these non-deformed planes. *2.* Velocity vectors are in each point tangential to the mentioned planes. For MR-flows a coordinate system can be *a priori* chosen so that the only non-vanishing component is

$$v_2 = \dot{x}_2 = v_2(x_1)\,. \tag{5}$$

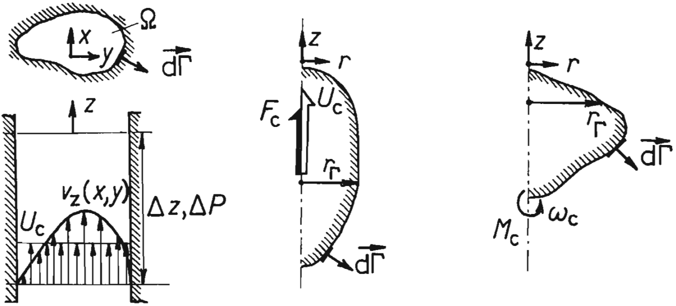
*3.* If *1.* and *2.* are fulfilled, the differential balance for the shear stress $\tau_{21} = \tau_{12}$ can be written in the form[15–17]

$$a = \frac{1}{h(x_1)} \cdot \frac{\mathrm{d}h(x_1)\,\tau_{12}}{\mathrm{d}x_1}\,, \tag{6}$$

where $a$ is the characteristic pressure gradient and $h(x_1)$ is a function,* dependent only on the metrics of the coordinate system $x_i$[15]. For MR-flows this function is of the power type (Table I).

TABLE I

Operating Variables in Some Typical Hydrodynamic Situations




| Flow through channels | Translation of bodies | Rotation of bodies |
|---|---|---|
| | Macroscopic momentum balance | |
| $\Delta P\,.\,\Omega = \Delta z \oint_\Gamma \mathbf{e}_z\,.\,\boldsymbol{\tau}\,.\,\mathrm{d}\Gamma$ | $F_c = 2\pi \int_\Gamma r_\Gamma \mathbf{e}_z\,.\,\boldsymbol{\tau}\,.\,\mathrm{d}\Gamma$ | $M_c = 2\pi \int_\Gamma r_\Gamma^2 \mathbf{e}_\varphi\,.\,\boldsymbol{\tau}\,.\,\mathrm{d}\Gamma$ |
| | $P_c$ mean integral value of deformation stress on the wall | |
| $P_c = (\Delta P/\Delta z)(\Omega/\Gamma)$ | $P_c = F_c/(2\pi\zeta R_c^2)$ | $P_c = M_c/(2\pi\zeta R_c^3)$ |
| | $R_c$ generalized conception of hydraulic diameter | |
| $R_c = \Omega/\Gamma$ | $R_c = \int_\Gamma r_\Gamma\,\mathrm{d}\Gamma/\Gamma$ | $R_c = (\int_\Gamma r_\Gamma^2\,\mathrm{d}\Gamma)/\Gamma^2$ |
| | $\zeta = \Gamma/R_c$ | $\zeta = \Gamma/R_c$ |
| | $U_c$ according to kinematic character of the problem | |
| $U_c = 1/\Omega \iint_\Omega v_z(x, y)\,\mathrm{d}\Omega$ | | $U_c = \omega_c R_c$ |

* Function $h(x_1)$ and further used functions $g(x_1)$ and $\varepsilon x_1^\gamma$ are called the metric coefficients. They are functions of the coordinates which appear in derivatives of tensor, vector, and scalar quantities along the curvilinear coordinates. The systematic study of these metric coefficients for general curvilinear geometrical coordinates is based on the conception of Rieman–Christoffel symbols[15].

$$h(x_1) \equiv kx_1^\alpha\,. \tag{7}$$

On the wall of the system, which is given by the relation $x_1 = x_{1w}$, the relation $\tau_{12}(x_{1w}) = \tau_w$ holds. Eq. (6) can be therefore integrated for $h(x_1)$ according to (7) in general to the form:

$$\tau_{12}/\tau_w = (1 - \nu)\,\xi + \nu\xi^{-\alpha}\,, \tag{8}$$

where

$$\xi = x_1/x_{1w}\,, \tag{9}$$

and where $\nu$ is an integration constant whose value for MR-flows depends only on the geometrical arrangement and which for MR-flows is equal either to 1 or 0.

For external flows when the solid body is surrounded by an infinitely large liquid volume, for $\xi \to \infty$, $\tau_{12} = 0$ must be valid, *i.e.* $\nu = 1$. For internal problems when the liquid flows due to the effect of a pressure drop through the space surrounded by stationary walls, a point of maximum velocity $\xi = 0$, $(\tau_{12} = 0)$ exists in the liquid so that now must be either $\nu = 0$, or $\alpha = 0$ (the latter is valid only for the simple shear flow).

Metzner–Rabinowitsch relations can now be derived under the above given assumptions. If the metric coefficients $h(x_1)$ are power functions, the metric coefficients $q(x_1)$ in the expression for the non-vanishing component of the shear rate tensor $D_{12}$, are power functions as well:

$$D_{12} = \frac{1}{g(x_1)} \cdot \frac{\mathrm{d}v_2(x_1)\, g(x_1)}{\mathrm{d}x_1} = x_1^\beta\, \frac{\mathrm{d}v_2(x_1)\, x_1^{-\beta}}{\mathrm{d}x_1}\,.$$

By introducing the normalized coordinate $\xi$ according to Eq. (9), $D_{12}$ can be expressed in the form

$$D_{12} = -\frac{1}{x_{1w}} \cdot \xi^\beta \cdot \frac{\mathrm{d}v_2(\xi)\, \xi^{-\beta}}{\mathrm{d}\xi}\,. \tag{10}$$

MR-flows are in agreement with the assumptions viscometric, therefore there exists a unique relation between $\tau_{12}$ and $D_{12}$,

$$-\frac{1}{x_{1w}} \cdot \xi^\beta \cdot \frac{\mathrm{d}v_2(\xi)\, \xi^{-\beta}}{\mathrm{d}\xi} = D[\tau_w((1 - \nu)\,\xi + \nu\xi^{-\alpha})]\,, \tag{11}$$

where $D[\tau]$ is the viscosity characteristics, a material function of the flowing non-Newtonian liquid[7]. According to (11) the velocity field can be expressed in the form

$$v_2(\xi) = x_{1w}\xi^\beta \int_\xi^\infty \xi^{-\beta} D[\tau_w \xi^{-\alpha}]\, \mathrm{d}\xi \tag{12a}$$

for internal flows, resp. in the form

$$v_2(\xi) = x_{1w}\xi^\beta \int_\xi^1 \xi^{-\beta} D[\tau_w \xi]\, \mathrm{d}\xi \tag{12b}$$

for external flows. In accordance with the above given relations the operational parameters $P_c$, $R_c$ can be chosen so that

$$P_c = \tau_w , \quad R_c = x_{1w} . \tag{13, 14}$$

However, only for external flows the characteristic velocity can directly be chosen as a velocity of the body in respect to the zero velocity of the liquid medium in infinity:

$$U_c = v_2(x_{1w}) = R_c \int_1^\infty \xi^{-\beta} D[\tau_w \xi^{-\alpha}] \, d\xi . \tag{15a}$$

For internal flows $U_c$ is introduced as the mean flow velocity

$$U_c = \frac{Q}{\Omega} = \frac{1}{\Omega} \iint_\Omega v_2(x_1) \, d\Omega(x_1, x_3) , \tag{16}$$

where the cross section $\Omega$ intersects all streamlines ($x_1$ = const; $x_3$ = const.). Due to the power type form of all metric coefficients in MR-flows, $U_c$ can be expressed, according to Eq. (*16*), in the form

$$U_c = \frac{\varepsilon}{R_c^\gamma} \int_0^{R_c} v_2(x_1) \, x_1^{\gamma-1} \, dx_1 = \varepsilon \int_0^1 v_2(\xi) \, \xi^{\gamma-1} \, d\xi , \tag{17}$$

where $\varepsilon$ and $\gamma$ are again parameters of a certain metric coefficient. Including the condition

$$v_2(x_{1w}) = 0 \tag{18}$$

and relations (*11*), (*12b*), the relation (*17*) can be, by integration *per partes*, modified into the final form

$$U_c = R_c \int_0^1 \frac{\varepsilon \xi^\gamma}{\gamma + \beta} \, D[\tau_w \xi] \, d\xi . \tag{15b}$$

By substituting the shear stress $\tau$ according to relations

$$\tau = \tau_w \xi^{-\alpha} , \tag{19a}$$

for external flows, resp.

$$\tau = \tau_w \xi , \tag{19b}$$

for internal flows, into relations (*15a*), resp. (*15b*), we obtain relations

$$\frac{U_c}{R_c} = \frac{\tau_w^{(1-\beta)/\alpha}}{\alpha} \int_0^{\tau_w} D[\tau] \, \tau^{(\beta-1)/\alpha} \, d\tau ; \quad \nu = 1 \tag{20a}$$

for external flows, resp.

$$\frac{U_c}{R_c} = \varepsilon \frac{\tau_w^{-(\gamma+1)}}{\gamma + \beta} \int_0^{\tau_w} D[\tau] \, \tau^\gamma \, d\tau \,, \tag{20b}$$

for internal flows. By derivation of these relations with respect to parameter $\tau_w$, by modification and substitution of $P_c = \tau_w$, we obtain a general expression of the Metzner–Rabinowitsch relations for the external flows

$$D_w = D[P_c] = \frac{U_c}{R_c} \cdot \left(\beta - 1 + \frac{\alpha}{n^*}\right), \tag{21a}$$

resp. for internal flows

$$D_w = D[P_c] = \frac{U_c}{R_c} \cdot \frac{\gamma + \beta}{\varepsilon} \cdot \left(\gamma + 1 + \frac{1}{n^*}\right), \tag{21b}$$

the parameter $n^*$ being defined by Eq. (*4*).

Examples for the above formulated general procedure are given in Table II, where corresponding relations for the Poiseuille flow, for the axial translation of a cylinder in an infinite liquid, and for the rotation of a cylinder in an infinite liquid are summarized.

Other well-know cases[7–9] of slit-flow, steady flow in a film, and of the simple shear flow, are not further considered.

## DIMENSIONLESS FLOW MODEL

For rheologically similar liquids[14] their viscosity characteristics are also similar, *i.e.* parameters $\tau_I$ and $D_I$ can be found so that their dimensionless characteristics defined by relation

$$p[\vartheta] \equiv D[\vartheta \tau_I]/D_I \text{ or } D[\tau] = D_I p[\tau/\tau_I] \,, \tag{22a, b}$$

are identical. If we introduce into the mathematical model of MR-flows, considered in previous paragraphs, two dimensionless numbers,

$$\mathrm{B} = (U_c/R_c)/D_I \,, \quad \mathrm{A} = P_c/\tau_I \tag{23, 24}$$

and the dimensionless viscosity characteristics according to Eq. (*22*), we can write the macroscopic momentum balances (*20a*), resp. (*20b*) in the form

$$\mathrm{B(A)} = \frac{\mathrm{A}^{(1-\beta)/\alpha}}{\alpha} \cdot \int_0^{\mathrm{A}} p[\vartheta] \, \vartheta^{[(\beta-1)/\alpha]-1} \, . \, d\vartheta \,, \tag{25a}$$

for external flows, resp.

$$\mathrm{B(A)} = \frac{\varepsilon^\beta \, \mathrm{A}^{-(\gamma+1)}}{\gamma + \beta} \cdot \int_0^{\mathrm{A}} p[\vartheta] \, \vartheta^\gamma \, d\vartheta \,, \tag{25b}$$

for internal flows.

The normalized velocity field for external flows can be written in the form

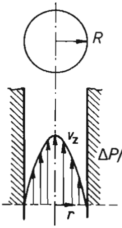
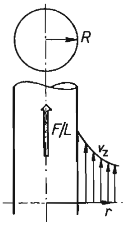
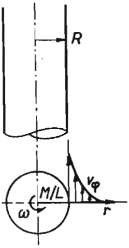
$$w(\xi, \mathrm{A}) = \frac{\xi^{\beta}\, \mathrm{A}^{(1-\beta)/\alpha}}{\mathrm{B}(\mathrm{A})} \cdot \int_0^{\mathrm{A}\,\xi^{-\alpha}} p[\vartheta]\, \vartheta^{[(\beta-1)/\alpha]-1}\, \mathrm{d}\vartheta\,, \tag{26a}$$

or

$$w(\xi, \mathrm{A}) = \frac{\mathrm{A}^{\beta-1}\xi^{\beta}}{\mathrm{B}(\mathrm{A})} \cdot \int_{\mathrm{A}\xi}^{\mathrm{A}} p[\vartheta]\, \vartheta^{-\beta}\, \mathrm{d}\vartheta\,, \tag{26b}$$

TABLE II

Basic Relations of Mathematical Models of Some MR-Flows

| Poiseuille flow | Translation of cylinder | Rotation of cylinder |
|---|---|---|
|  |  |  |
| $P_c = \tau_w = \frac{\Delta P}{\Delta Z} \cdot \frac{R}{2}$ | $P_c = \tau_w = \frac{F}{2\pi RL}$ | $P_c = \tau_w = \frac{M}{2\pi LR^2}$ |
| $R_c = R$ | $R_c = R$ | $R_c = R$ |
| $U_c = \frac{Q}{\pi R^2} = \frac{2}{R^2} \cdot \int_0^R v_z(r)\, r\, \mathrm{d}r$ | $U_c = v_z(R)$ | $U_c = v_\varphi(R) = \omega R$ |
| $\tau_{rz} = \tau_w \cdot \left(\frac{r}{R}\right)$ | $\tau_{rz} = \tau_w \cdot \frac{R}{r}$ | $\tau_{r\varphi} = \tau_w \cdot \left(\frac{R}{r}\right)^2$ |
| $\frac{\Delta P}{\Delta z} = \frac{1}{r} \cdot \frac{\mathrm{d}(r\tau_{rz})}{\mathrm{d}r}$ | $0 = \frac{1}{r} \cdot \frac{\mathrm{d}(r\tau_{rz})}{\mathrm{d}r}$ | $0 = \frac{1}{r^2} \cdot \frac{\mathrm{d}(v^2 \,.\, \tau_{r\varphi})}{\mathrm{d}r}$ |
| $D_{rz} = -\frac{\mathrm{d}v_z}{\mathrm{d}r}$ | $D_{rz} = -\frac{\mathrm{d}r_z}{\mathrm{d}r}$ | $D_{r\varphi} = -r \,.\, \frac{\mathrm{d}v_\varphi}{\mathrm{d}r}$ |
| $\nu = 0,\ \beta = 0,\ \gamma = 2,\ \varepsilon = 2$ | $\nu = 1,\ \alpha = 1,\ \beta = 0$ | $\nu = 1,\ \alpha = 2,\ \beta = 0$ |
| $D_w \frac{R_c}{U_c} = 3 + \frac{1}{n^*}$ | $D_w \frac{R_c}{U_c} = \frac{1}{n^*} - 1$ | $D_w \frac{R_c}{U_c} = \frac{2}{n^*}$ |

for the internal flows. The dimensionless velocity $w(\xi)$ is defined by relation

$$w(\xi) = v_2(\xi x_{1w})/U_c\,. \tag{27}$$

According to the definitions of respective dimensionless quantities, the course of normalized shear stresses is obviously given by relations

$$\tau/\tau_I = \vartheta(\xi, A) = \begin{cases} A\xi^{-\alpha} & \text{(external)} \quad (28a) \\ A\xi & \text{(internal)} \quad (28b) \end{cases}$$

The course of normalized shear rates can be thus written as

$$D/D_I = p[\vartheta] = \begin{cases} p[A\xi^{-\alpha}] & \text{(external)} \quad (29a) \\ p[A\xi] & \text{(internal)} \quad (29b) \end{cases}$$

and their values on the wall are given by the relation

$$p_w = D_w/D_I = p[A]\,. \tag{30}$$

Since $D_I$ and $\tau_I$ are constant for the same liquid, we can express $n^*$ defined by relation (4) in the form

$$n^* = d \ln A/d \ln B\,. \tag{31}$$

Since according to relations (22)

$$\frac{R_c}{U_c} = \frac{p[A]}{B(A)} = D_w\,.\,\Phi(n^*)\,, \tag{32}$$

it is possible to express $n^*$ from relations (21a, b) for a known course of the dimensionless viscosity characteristics $p[\vartheta]$ and for values of B[A] according to relations (26a), (26b) explicitly as:

$$1/n^* = \psi(A, p[\vartheta]) = \begin{cases} \dfrac{p[A]\,A^{(\beta-1)/\alpha}}{\int_0^A p[\vartheta]\,\vartheta^{[(\beta-1)/\alpha]-1}\,d\vartheta} - (\beta - 1) & \text{(external)} \quad (33a) \\ \dfrac{p[A]\,A^{\gamma+1}}{\int_0^A p[\vartheta]\,\vartheta^{\gamma}\,d\vartheta} - (\gamma + 1) & \text{(internal)} \quad (33b) \end{cases}$$

## Approximate Kinematic Similarity of MR-Flows

The kinematic similarity of two flows is understood by us as the possibility to normalize the velocity fields $v_2(x_1)$ by suitably introduced parameters $U_c$, $R_c$ so that the normalized velocity fields $w(\xi)$ have identical courses. From dimensionless relations presented in the previous chapter is obvious that the indispensable and sufficient condition for the kinematic similarity of two MR-flows, $w(\xi) = \text{idem}$, is the condition $\mathrm{A} = \text{idem}$ or $\mathrm{B} = \text{idem}$, and the condition of rheological similarity in the form $p[\vartheta] \equiv \text{idem}$.

These are relatively pretentious conditions which in fact eliminate the possibility to achieve a kinematic flow similarity of two non-Newtonian liquids because it is highly improbable that the course of functions $p[\vartheta]$ could be fully identical for them in the given interval $\vartheta \in \langle 0; \mathrm{A} \rangle$. In this context it appears desirable to find other less pretentious criteria which would ensure at least an approximate kinematic similarity, *i.e.* approximately the same course of $w(\xi)$ for flows of two different non-Newtonian liquids.

Let us start with the definition *(27)* of the dimensionless velocity: by its use *i.e.* according to relation *(11)* the courses of deformation velocities are expressed as

$$p[\vartheta] = \mathrm{B}\,\xi^{\beta}\,\frac{\mathrm{d}w(\xi)\,\xi^{-\beta}}{\mathrm{d}\xi}\,. \tag{34}$$

According to relation *(34)* the normalized velocity gradient on the wall for $\vartheta = \mathrm{A}$, *i.e.* for $\xi \doteq 1$ can be expressed in the form

TABLE III

Basic Relations in Dimensionless Mathematical Models of MR-flows

| Poiseuille flow | Translation of cylinder | Rotation of cylinder |
|---|---|---|
| $\xi = r/R$ | $\xi = r/R$ | $\xi = r/R$ |
| $\mathrm{A} = P_c/\tau_I$ | $\mathrm{A} = P_c/\tau_I$ | $\mathrm{A} = P_c/\tau_I$ |
| $\vartheta = \mathrm{A}\,\xi$ | $\vartheta = \mathrm{A}\,\xi^{-1}$ | $\vartheta = \mathrm{A}\,\xi^{-2}$ |
| $-\dfrac{\mathrm{d}w}{\mathrm{d}\xi} = \dfrac{p[\vartheta]}{\mathrm{B}}$ | $-\dfrac{\mathrm{d}w}{\mathrm{d}\xi} = \dfrac{p[\vartheta]}{\mathrm{B}}$ | $-\xi\dfrac{\mathrm{d}w/\xi}{\mathrm{d}\xi} = \dfrac{p[\vartheta]}{\mathrm{B}}$ |
| $\mathrm{B}[\mathrm{A}] = \dfrac{1}{\mathrm{A}^3}\displaystyle\int_0^{\mathrm{A}} p[\vartheta]\,\vartheta^2\,\mathrm{d}\vartheta$ | $\mathrm{B}[\mathrm{A}] = \mathrm{A}\displaystyle\int_0^{\mathrm{A}} = p[\vartheta]\,\frac{\mathrm{d}\vartheta}{\vartheta^2}$ | $\mathrm{B}[\mathrm{A}] = \dfrac{1}{2}\displaystyle\int_0^{\mathrm{A}} p[\vartheta]\,\frac{\mathrm{d}\vartheta}{\vartheta}$ |
| $w(\xi) = \dfrac{1}{\mathrm{BA}}\displaystyle\int_{\mathrm{A}\xi}^{\mathrm{A}} p[\vartheta]\,\mathrm{d}\vartheta$ | $w(\xi) = \dfrac{\mathrm{A}}{\mathrm{B}}\displaystyle\int_0^{\mathrm{A}\xi^{-1}} p[\vartheta]\,\frac{\mathrm{d}\vartheta}{\vartheta^2}$ | $w(\xi) = \dfrac{\xi}{2\mathrm{B}}\displaystyle\int_0^{\mathrm{A}\xi^{-2}} p[\vartheta]\,\frac{\mathrm{d}\vartheta}{\vartheta}$ |
| $n^*(\mathrm{A}) = \dfrac{1}{\dfrac{p[\mathrm{A}]}{\mathrm{B}[\mathrm{A}]} - 3}$ | $n^*(\mathrm{A}) = \dfrac{1}{\dfrac{p[\mathrm{A}]}{\mathrm{B}[\mathrm{A}]} + 1}$ | $n^*(\mathrm{A}) = \dfrac{2\mathrm{B}[\mathrm{A}]}{p[\mathrm{A}]}$ |

$$-\left.\frac{\mathrm{d}w}{\mathrm{d}\xi}\right|_1 + \beta\, w(1) = \varphi(n^*)\,, \tag{35}$$

where $\varphi(n^*)$ is given by relations (*21a*) resp. (*21b*) in general and by relations given in Table III for individual earlier discussed MR-flows.

Function $w(\xi)$ is thus bound on the one hand by the normalizing conditions

$$w(1) = 1\,, \qquad \text{(external)} \tag{36a}$$

$$1 = \varepsilon \int_0^1 w(\xi)\, \xi^{\gamma - 1}\, \mathrm{d}\xi\,, \qquad \text{(internal)} \tag{36b}$$

and on the other hand by relation (*35*) if $n^*$ is considered to be a fixed parameter. Furthermore, we may intuitively expect that the function $w(\xi)$ regardless of differences in the course of viscosity characteristics will not have any further extremes, inflex points *etc.* in comparison with the course $w(\xi)$ for Newtonian liquids.

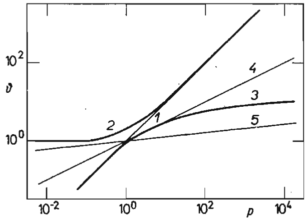



FIG. 1

Courses of Viscosity Characteristics According to Some Models

$p$ Normalized shear rate, $\vartheta$ normalized shear stress, *1* Newton model, *2* Bingham model, *3* Eyring model, *4* power-law model, $n = 0{\cdot}5$, *5* power-law model, $n = 0{\cdot}2$.

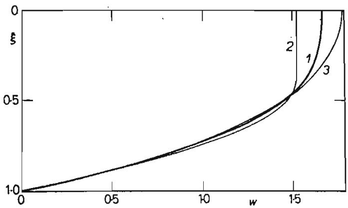



FIG. 2

Courses of Normalized Velocity Profiles for Different Models for Pipe Flow, $n^* = 0{\cdot}5$
*1* Power-law model, *2* Bingham model, *3* Eyring model.

We may expect, however, various smaller deviations in the courses of $w(\xi)$ since for different $p(\vartheta)$ the relations (*26a,b*) have different forms, too. In practical applications the course of measured viscosity characteristics is interpreted by rheological models and by various empirical or semi-empirical correlation relations. Among the most frequently used belongs

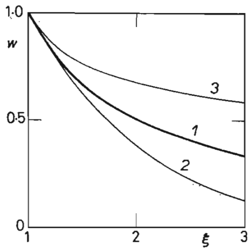



FIG. 3

Courses of Normalized Velocity Profiles for Flow around a Towed Cylinder, $n^* = 0{\cdot}5$

*1* Power-law model, *2* Bingham model, *3* Eyring model.

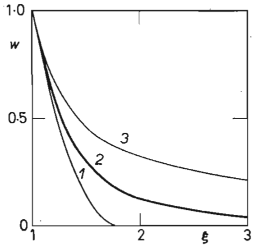



FIG. 4

Courses of Normalized Velocity Profiles for Flow around a Rotating Cylinder, $n^* = 0{\cdot}5$

*1* Power-law model, *2* Bingham model, *3* Eyring model.

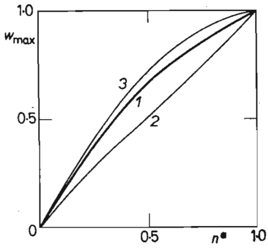



FIG. 5

Dependence of Normalized Maximum Velocity for Pipe Flow on Apparent Flow Index $n^*$

*1* Power-law model, *2* Bingham model, *3* Eyring model.

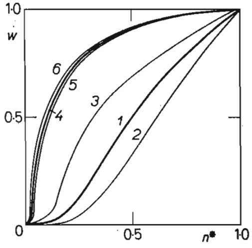



FIG. 6

Characteristic Normalized Velocity in Liquid Flowing around the Cylinder in Axial Motion

Distance of axis from cylinder $\xi = 2{\cdot}0$ (*1*–*3*) resp. $\xi = 1{\cdot}1$ (*4*–*6*); *1*, *4* power-law model, *2*, *5* Bingham model, *3*, *6* Eyring model.

the power-law model

$$D = (\tau/K)^{1/n}\,, \tag{37a}$$

resp. in the dimensionless form

$$p = \vartheta^{1/n}\,, \tag{37b}$$

the Bingham model

$$D = (\tau - \tau_\mathrm{I})/\mu_\mathrm{I}\,, \tag{38a}$$

resp., in the dimensionless form

$$p = \vartheta - 1\,, \tag{38b}$$

and the Eyring model

$$D = D_\mathrm{I} \sinh(\tau/\tau_\mathrm{I})\,, \tag{39a}$$

resp., in the dimensionless form

$$p = \sinh \vartheta\,. \tag{39b}$$

In Fig. 1 the courses of these three models are plotted in coordinates $\log p - \log \vartheta$ (the power model includes furthermore the dimensionless simplex $n$ and is thus represented by a set of curves with the parameter $n$) and it is obvious that these courses substantially differ. Available experimental data on viscosity characteristics of various non-Newtonian materials[3-6] even indicate that with a great probability it can be expected that the Bingham model on the one side and the Eyring model on the other side represent the extreme cases of viscosity characteristics which can be expected with actual materials.*

In Table III the most important dimensionless calculation relations are given for three MR-flows described in Fig. 1. In Table IV the relations for the mentioned three models are evaluated. In Fig. 2–4 the corresponding normalized velocity profiles are plotted for $n^* = 0{\cdot}5$ for the Poiseuille flow, the axial motion of a cylinder, and for the cylinder rotation, respectively.

From Fig. 2 it follows that the largest deviations in the course of the normalized velocity profiles in the Poiseuille flow occur in the point $\xi = 0$, thus these deviations can be represented by values of normalized maximum velocities $w_{max} = w(0)$. In Fig. 5 the values of $w_{max}$ for Poiseuille flow for the three discussed models of viscosity characteristics are therefore plotted in dependence on $n^*$.

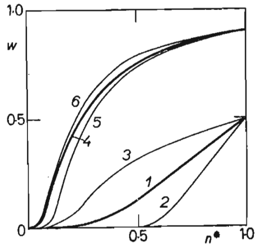



FIG. 7

Characteristic Normalized Velocities in Liquid Flowing Around a Rotating Cylinder

Distance from cylinder axis $\xi = 2{\cdot}0$ (*1–3*) resp. $\xi = 1{\cdot}1$ (*4–6*); *1*, *4* power-law model, *2*, *5* Bingham model, *3*, *6* Eyring model.

* The viscosity characteristics of actual materials are usually more straight than the two mentioned extreme models. By a suitable choice of $D_\mathrm{I}$ and $\tau_\mathrm{I}$, resp. by suitable shifting of the experimentally determined curve in Fig. 1 it can be achieved that the whole curve is situated in the area limited in Fig. 1 by the curves for the Eyring and Bingham models.

TABLE IV

MR-Flows for Empirical Models of Viscosity Characteristics

| | Flow in pipe | Translation of cylinder | Rotation of cylinder |
|---|---|---|---|
| Power-law model<br>$p[\vartheta] = \vartheta^{1/n}$ | $B = \dfrac{A^{1/n}}{1/n + 3}$<br>$w = \dfrac{1+3n}{1+n}\left(1 - \xi^{\frac{1+n}{n}}\right)$ | $B = \dfrac{A^{1/n}}{1/n - 1}$<br>$w = \xi^{-\frac{1-n}{n}}$ | $B = \dfrac{n}{2} \cdot A^{1/n}$<br>$w = \xi^{-\frac{2-n}{n}}$ |
| Bingham model<br>$p[\vartheta] = \begin{cases} \vartheta - 1\,; & \vartheta > 1 \\ 0\,; & \vartheta < 1 \end{cases}$ | $B = \dfrac{(A-1)^2}{A^3}\dfrac{3A^2 + 2A + 1}{12}$<br>$w = \dfrac{2(1-\xi^2) - 4(1-\xi)/A}{1 - A^{-1}/3 + A^{-4}/12}$<br>$n^* = \dfrac{(A-1)^2\,(3A^2 + 2A + 1)}{3(A^2 - 1)}$ | $B = A(\ln A - 1) + 1$<br>$w = \dfrac{\ln(A/\xi) + (\xi/A) - 1}{\ln A + (1/A) - 1}$<br>$n^* = 1 + \dfrac{1 - A}{A \ln A}$ | $B = 12(A - 1 - \ln A)$<br>$w = \dfrac{A/\xi - \xi(\ln(A/\xi^2) + 1)}{A - (\ln A + 1)}$<br>$n^* = 1 - \dfrac{\ln A}{A - 1}$ |
| Eyring model<br>$p[\vartheta] = \sinh(\vartheta)$ | $B = \{c - 2[s - (c-1)/A]/A\}/A$<br>$w = \dfrac{C - \cosh(A\xi)}{AB}$<br>$n^* = 1\Big/\left[\dfrac{\sinh(A)}{B} - 3\right]$<br>where<br>$c = \cosh(A)$<br>$s = \sinh(A)$ | (no exact solution)<br>$B \approx AG(A)$<br>$w(\xi) \approx G(A/\vartheta)/G(A)$<br>$n^* \approx 1\Big/\left[\dfrac{\sinh(A)}{AG(A)} + 1\right]$<br>where<br>$G(x) = \displaystyle\int_{0.01}^{x} \frac{\sinh(\vartheta)}{\vartheta^2}\,d\vartheta$ | $B = \mathrm{shi}(A)$<br>$w = \xi\,\dfrac{\mathrm{shi}(A/\xi^2)}{\mathrm{shi}(A)}$<br>$n^* = \dfrac{\mathrm{shi}(A)}{\sinh(A)}$<br>where[17]<br>$\mathrm{shi}(x) \equiv [(E_i^*(x) - E_i(x)]/2$ |

An important practical aspect, with internal problems, for which the respective normalized velocity profiles are plotted in Figs 3, 4, is how quickly the velocity decreases toward zero with the distance from the walls of a moving cylinder. Figs 6, 7 therefore give the velocities $w(1,1)$ and $w(2)$ at distances $\xi = 1,1$ and $\xi = 2$ from the centre of the cylinder, in dependence on parameter $n^*$.

## CONCLUSION

It is obvious from Figs 5–7 where the dependence of parameters characterizing the course of a normalized velocity profile which are for a given $n^*$ also dependent on the overall course of the viscosity characteristics are plotted, that on the basis of known $n^*$, their values can be predicted with an accuracy $\pm 10\%$ as the value of these parameters for power-law model. From Figs 2–4 is obvious that with the use of known $n^*$ roughly with the same accuracy the overall courses of normalized velocity profiles can be also estimated. As follows from Figs 2–7 for the course of normalized velocity profiles and the relevant parameters the scatter for real non-Newtonian liquids will be always rather smaller. This authorizes us to the at first sight bizarre statement valid for MR-flows: "On the basis of measurements of two values $P_c$ for two different but not too differing values $U_c$ at a given flow arrangement the course of the velocity field in this flow situation can be roughly estimated".

*The authors wish to express their thanks to Miss S. Nováková for doing the numerical work and drawing the figures.*

### LIST OF SYMBOLS

| | |
|---|---|
| $a$ | pressure gradient (dyn cm$^{-3}$) |
| $F$ | force (dyn) |
| $D_{ij}$ | components of shear rate tensor (s$^{-1}$) |
| $D$ | scalar shear rate, second invariant of tensor $D_{ij}$ (s$^{-1}$) |
| $D_I$ | material constant (s$^{-1}$) |
| $D_w$ | shear rate on the wall (s$^{-1}$) |
| $K$ | viscosity coefficient (dyn cm$^{-2}$ s$^n$) |
| $L$ | length of cylinder or pipe to which the values are related (cm) |
| $M$ | couple (dyn cm) |
| $n$ | flow index (1) |
| $n^*$ | apparent flow index (1) |
| $P$ | hydrodynamic potential, pressure (dyn cm$^{-2}$) |
| $p$ | shear rate, dimensionless |
| $P_c$ | characteristic stress (dyn cm$^{-2}$) |
| $\Delta P$ | pressure drop (dyn cm$^{-2}$) |
| $Q$ | volumetric flow rate (cm$^3$ s$^{-1}$) |
| $r$ | radial coordinate (cm) |
| $R$ | radius of cylinder (pipe) (cm) |
| $R_c$ | characteristic length (cm) |
| $U_c$ | characteristic velocity (cm s$^{-1}$) |
| $v_z$ | axial velocity (cm s$^{-1}$) |

| | |
|---|---|
| $v_\theta$ | tangential velocity (cm $s^{-1}$) |
| $w$ | normalized velocity (1) |
| $x_i$ | geometrical coordinates (cm) |
| $x_{1w}$ | value of coordinate $x_1$ on the wall (cm) |
| $\alpha, \beta, \gamma, \varepsilon, \nu$ | parameters of MR-flow models (1) |
| $\omega$ | angular velocity ($s^{-1}$) |
| $\Omega$ | cross section ($cm^2$) |
| $\xi$ | dimensionless coordinate $x_1$ (1) |
| $\vartheta$ | shear stress, dimensionless (1) |
| $\tau_{ij}$ | components of shear stress tensor (dyn $cm^{-2}$) |
| $\tau$ | scalar shear stress (dyn $cm^{-2}$) |
| $\tau_w$ | shear stress on the wall (dyn $cm^{-2}$) |
| $\tau_I$ | material constant (dyn $cm^{-2}$) |
| $A = P_c/\tau_I$ | dimensionless number |
| $B = (U_c/R_c)/D_I$ | dimensionless number |

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